Chapter III

Classification of partial differential equations into elliptic, parabolic and hyperbolic types

The previous chapters have displayed examples of partial differential equations in various fields of mathematical physics. Attention has been paid to the interpretation of these equations in the specific contexts they were presented. [1]

In fact, we have delineated three types of field equations, namely hyperbolic, parabolic and elliptic. The basic idea that the mathematical nature of these equations was fundamental to their physical significance has been creeping throughout.

Still, the formats in which these three types were presented correspond to their canonical forms, that is, a form that one recognizes at first glance. Such is not the general case. For example, it is not obvious (to this author at least!) that the following second order equation,

$$2\,\frac{\partial^2 u}{\partial x^2} - 4\,\frac{\partial^2 u}{\partial x \partial t} - 6\,\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x} = 0\,,$$

is of hyperbolic type. In other words, it shares essential physical properties with the wave equation,

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0\,.$$

Indeed, this is the aim of the present chapter to show that all equations of mathematical physics can be recast in these three fundamental types. By the same token, we introduce a new notion, that of a characteristic curve. A method to solve IBVPs based on characteristics will be exposed in the next chapter.

The terminology used to coin the three types of PDEs borrows from geometry, as the criterion will be seen to rely on the nature of the roots of quadratic equations.

We envisage in turn first of order equations, sets of first order equations, and second order equations. The use of a common terminology to class first and second order equations is challenged by the fact that a set of two first order equation may be transformed into a second

[1] Posted, December 05, 2008; updated, December 12, 2008

order equation, and conversely. The point will not be developed throughout, but rather treated via examples.

Since we are concerned in this chapter with the nature of partial differential equtions, we will not specify the domain in which they assume to hold. On the other hand, the issue surfaces when we intend to solve IBVPs, as considered in Chapters I, II and IV.

III.1 First order partial differential equations

III.1.1 A single equation

We consider first a single first order partial differential equation for the unknown function $u = u(x, y)$,

$$\begin{aligned} &u = u(x, y) \quad \text{unknown}, \\ &(x, y) \quad \text{variables}, \end{aligned} \tag{III.1.1}$$

that can be cast in the format,

$$a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + c = 0\,. \tag{III.1.2}$$

This equation is said to be (please think a little bit to this terminology),

- *linear* if $a = a(x, y)$, $b = b(x, y)$, and c constant;
- *quasi-linear* if these coefficients depend in addition on the unknown u;
- *nonlinear* if these coefficients depend further on the derivatives of the unknown u.

Let

$$\mathbf{s} = \frac{1}{\sqrt{a^2 + b^2}} \begin{bmatrix} a \\ b \end{bmatrix}, \tag{III.1.3}$$

be the unit vector that makes it possible to recast the PDE (III.1.2) into the format,

$$\mathbf{s} \cdot \nabla u + d = 0\,, \tag{III.1.4}$$

with $d = c/\sqrt{a^2 + b^2}$.

The curves, starting from an initial curve I_0, and with a slope,

$$\frac{dy}{dx} = \frac{b}{a}\,, \tag{III.1.5}$$

are called **characteristic curves**. A point on these curves is reckoned by the curvilinear abscissa σ,

$$(d\sigma)^2 = (dx)^2 + (dy)^2\,. \tag{III.1.6}$$

Typically, σ is set to 0 on the initial curve I_0.

Then

$$\mathbf{s} = \begin{bmatrix} dx/d\sigma \\ dy/d\sigma \end{bmatrix}, \tag{III.1.7}$$

and **the partial differential equation (PDE)** (III.1.4) for $u(x, y)$,

$$\frac{\partial u}{\partial x} \frac{dx}{d\sigma} + \frac{\partial u}{\partial y} \frac{dy}{d\sigma} + d = \frac{du}{d\sigma} + d = 0\,, \tag{III.1.8}$$

magically becomes **an ordinary differential equation (ODE)** for $u(\sigma)$ along a characteristic $dy/dx = b/a$. Hum$\cdots$ puzzling, how is that possible? There should be a trick here $\cdots$ My mum warned me, "my little boy, nothing comes for free in this world, except AIDS perhaps". Indeed, there is a price to pay, and the price is to find the characteristic curves, which are not known beforehand.

Taking a step backward, the transformation of a PDE to an ODE is a phenomenon that we have already encountered. Indeed, this is in fact the basic principle of Laplace or Fourier transforms. The initial PDE is transformed into an ODE where the variable associated to the transform is temporarily seen as a parameter. The price to pay here is the inverse transformation.

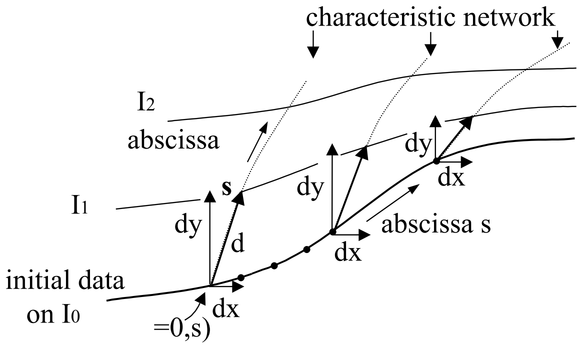

Figure III.1 Given data on a non characteristic initial curve I_0, the characteristic network and solution are built simultaneously, step by step. Each characteristic is endowed with a curvilinear abscissa σ, while points on the initial curve I_0 are reckoned by a curvilinear abscissa s.

Analytical and/or Numerical solution

The above observations provide the basics to a method for solving a partial differential equation.

If the PDE is linear, then

- the characteristics and curvilinear abscissa are obtained by (III.1.5) and (III.1.6);
- the solution u is deduced from (III.1.8).

If the PDE is quasi-linear, a numerical scheme is developed to solve simultaneously (III.1.5) and (III.1.8):

- assume u to be known along a curve I_0, which is required not to be a characteristic;
- at each point of I_0, one may obtain and draw the characteristic using (III.1.5), which provides also $d\sigma$ by (III.1.6);
- du results from (III.1.8), whence the solution on the new curve I_1;
- the three steps above are repeated, starting from I_1, and so on.

It is now clear why the initial curve I_0 should not be a characteristic. Indeed, otherwise, the subsequent curves $I_1 \cdots$ would be I_0 itself, so that the solution could not be obtained at points (x, y) other than on I_0.

III.1.2 A system of quasi-linear equations

The concept of a characteristic curve is now extended to a quasi-linear system of first order partial differential equations for the n unknown functions $u's$,

$$\begin{aligned} &u_j = u_j(x,t), \quad j \in [1,n], \quad \text{unknowns}, \\ &(x,t) \quad \text{variables}\,, \end{aligned} \tag{III.1.9}$$

that can be cast in the format,

$$\begin{aligned} \mathcal{L}\cdot\mathbf{U} &= \mathbf{a}\cdot\frac{\partial\mathbf{u}}{\partial t} + \mathbf{b}\cdot\frac{\partial\mathbf{u}}{\partial x} + \mathbf{c} = \mathbf{0} \\ \mathcal{L}_{ij}\,u_j &= a_{ij}\,\frac{\partial u_j}{\partial t} + b_{ij}\,\frac{\partial u_j}{\partial x} + c_i = 0\,, \quad i\in[1,n]\,, \end{aligned} \tag{III.1.10}$$

where the coefficient matrices $\mathbf{a}=(a_{ij})$ and $\mathbf{b}=(b_{ij})$ with $(i,j)\in[1,n]^2$, and vector $\mathbf{c}=(c_i)$, with $i\in[1,n]$, may depend on the variables and unknowns, but not of their derivatives.

In order to form an *ordinary* differential equation in terms of a (yet) unknown curvilinear abscissa σ, we devise a linear combination of these n partial differential equations, namely,

$$\begin{aligned} \boldsymbol{\lambda}\cdot\mathcal{L}\cdot\mathbf{u} &= \mathbf{p}\cdot\frac{d\mathbf{u}}{d\sigma} + r = 0 \\ \lambda_i\mathcal{L}_{ij}\,u_j &= p_j\,\frac{du_j}{d\sigma} + r = 0\,. \end{aligned} \tag{III.1.11}$$

The vector $\boldsymbol{\lambda}$ will appear to be a left eigenvector of the matrix $\mathbf{a}\,dx/dt-\mathbf{b}$, namely

$$\begin{aligned} \boldsymbol{\lambda}\cdot(\mathbf{a}\frac{dx}{dt}-\mathbf{b}) &= \mathbf{0}, \\ \lambda_i\;(a_{ij}\frac{dx}{dt}-b_{ij}) &= 0, \quad j\in[1,n]\,. \end{aligned} \tag{III.1.12}$$

To prove this property, we pre-multiply (III.1.10) by $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda}\cdot\mathbf{a}\cdot\frac{\partial\mathbf{u}}{\partial t} + \boldsymbol{\lambda}\cdot\mathbf{b}\cdot\frac{\partial\mathbf{u}}{\partial x} + \boldsymbol{\lambda}\cdot\mathbf{c} = 0\,, \tag{III.1.13}$$

which can be of the form (III.1.11) only if

$$\frac{\boldsymbol{\lambda}\cdot\mathbf{a}}{dt} = \frac{\boldsymbol{\lambda}\cdot\mathbf{b}}{dx} = \frac{\mathbf{p}}{d\sigma}\,. \tag{III.1.14}$$

Elimination of the vector $\mathbf{p}$ in this relation yields the generalized eigenvalue problem (III.1.12). For the eigenvector $\boldsymbol{\lambda}$ not to vanish, the associated coefficient matrix should be singular,

$$\det\,(\mathbf{a}\frac{dx}{dt}-\mathbf{b}) = 0\,. \tag{III.1.15}$$

This characteristic equation should be seen as a polynomial equation of degree n for dx/dt. The classification of first order partial differential equations is based on the above spectral analysis.

Classification of first order linear PDEs

- if the nb of real eigenvalues is 0, the system is said **elliptic**;
- if the eigenvalues are real and distinct, or
  if the eigenvalues are real and the system is not defective, the system is said **hyperbolic**;
- if the eigenvalues are real, but the system is defective, the system is said to be **parabolic**.

Let us recall that a system of size n is said non defective if its eigenvectors generate $\mathbb{R}^n$, that is, the algebraic and geometric multiplicities of each eigenvector are identical.

Characteristic curves and Riemann invariants

Each eigenvalue dx/dt defines a curve in the plane (x,t) called characteristic. To each characteristic is associated a curvilinear abscissa σ, defined by its differential,

$$\begin{aligned} \frac{d}{d\sigma} &= \frac{dt}{d\sigma}\,\frac{\partial}{\partial t} + \frac{dx}{d\sigma}\,\frac{\partial}{\partial x} \\ &= \frac{dt}{d\sigma}\Big(\frac{\partial}{\partial t} + \frac{dx}{dt}\,\frac{\partial}{\partial x}\Big)\,. \end{aligned} \tag{III.1.16}$$

Inserting (III.1.12) into (III.1.13) yields,

$$\boldsymbol{\lambda} \cdot \mathbf{a} \cdot \frac{d\mathbf{u}}{d\sigma} + \frac{dt}{d\sigma} \boldsymbol{\lambda} \cdot \mathbf{c} = 0 \,. \tag{III.1.17}$$

Quantities that are constant along a characteristic are called Riemann invariants.

A simple, but subtle and tricky issue

1. Please remind that the left and right eigenvalues of an arbitrary square matrix are identical, but the left and right eigenvectors do not, if the matrix is not symmetric. The left eigenvectors of a matrix $\mathbf{a}$ are the right eigenvectors of its transpose $\mathbf{a}^T$.
2. The generalized (left) eigenvalue problem $\boldsymbol{\lambda} \cdot (\mathbf{a}\, dx/dt - \mathbf{b}) = \mathbf{0}$ becomes a standard (left) eigenvalue problem when $\mathbf{b} = \mathbf{I}$, i.e. $\boldsymbol{\lambda} \cdot (\mathbf{a}\, dx/dt - \mathbf{I}) = \mathbf{0}$. The left eigenvectors of the pencil $(\mathbf{a}, \mathbf{b})$ are also the right eigenvectors of the pencil $(\mathbf{a}^T, \mathbf{b}^T)$.
3. Note the subtle interplay between the sets of matrices $(\mathbf{a}, \mathbf{b})$, and the variables (t, x). The above writing has made use of the ratio dx/dt, and not of dt/dx: we have broken symmetry without care. That temerity might not be without consequence. Indeed, an immediate question comes to mind: are the eigenvalue problems $\boldsymbol{\lambda} \cdot (\mathbf{a}\, dx/dt - \mathbf{b}) = \mathbf{0}$ and $\boldsymbol{\lambda} \cdot (\mathbf{a} - \mathbf{b}\, dt/dx) = \mathbf{0}$ equivalent? The answer is not so straightforward, as will be illustrated in Exercise III.2.

Some further terminology

If the system of PDEs,

$$\mathbf{a} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathbf{b} \cdot \frac{\partial \mathbf{u}}{\partial x} + \mathbf{c} = \mathbf{0} \,, \tag{III.1.18}$$

can be cast in the format,

$$\frac{\partial \mathbf{F}(\mathbf{u})}{\partial t} + \frac{\partial \mathbf{G}(\mathbf{u})}{\partial x} = \mathbf{0} \,, \tag{III.1.19}$$

it is said to be of **divergence type**. In the special case where the system can be cast in the format,

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{G}(\mathbf{u})}{\partial x} = \mathbf{0} \,, \tag{III.1.20}$$

it is termed a **conservation law**.

III.2 Second order partial differential equations

The analysis addresses a single equation, delineating the case of constant coefficients from that of variable coefficients.

III.2.1 A single equation with constant coefficients

Let us start with an example. For the homogeneous wave equation,

$$\mathcal{L}u = \frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0 \,, \tag{III.2.1}$$

the change of coordinates,

$$\xi = x - c\,t, \quad \eta = x + c\,t \,, \tag{III.2.2}$$

transforms the *canonical form* (III.2.1) into another *canonical form*,

$$\mathcal{L}u = \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \,. \tag{III.2.3}$$

Therefore, the solution expresses in terms of two arbitrary functions,

$$u(\xi,\eta) = f(\xi) + g(\eta)\,, \tag{III.2.4}$$

which should be prescribed along a non characteristic curve.

But where are the characteristics here? Well, simply, they are the lines ξ constant and η constant.

Let us try to generalize this result to a second order partial differential equation for the unknown $u(x,y)$,

$$\begin{aligned} &u = u(x,y) \quad \text{unknown,} \\ &(x,y) \quad \text{variables}\,, \end{aligned} \tag{III.2.5}$$

with constant coefficients,

$$\mathcal{L}u = A\,\frac{\partial^2 u}{\partial x^2} + 2\,B\,\frac{\partial^2 u}{\partial x \partial y} + C\,\frac{\partial^2 u}{\partial y^2} + D\,\frac{\partial u}{\partial x} + E\,\frac{\partial u}{\partial y} + F\,u + G = 0\,. \tag{III.2.6}$$

The question is the following: can we find characteristic curves, so as to cast this PDE into an ODE? The answer was positive for the wave equation. What do we get in this more general case?

Well, we are on a moving ground here. To be safe, we should keep some degrees of freedom. So we bet on a change of coordinates,

$$\xi = -\alpha_1\,x + y, \quad \eta = -\alpha_2\,x + y\,, \tag{III.2.7}$$

where the coefficients α_1 and α_1 are left free, that is, they are to be discovered.

Now come some tedious algebras,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = -\alpha_1\,\frac{\partial u}{\partial \xi} - \alpha_2\,\frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\,, \end{aligned} \tag{III.2.8}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \alpha_1^2\,\frac{\partial^2 u}{\partial \xi^2} + 2\,\alpha_1\,\alpha_2\,\frac{\partial^2 u}{\partial \xi \partial \eta} + \alpha_2^2\,\frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2} + 2\,\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial x \partial y} &= -\alpha_1\,\frac{\partial^2 u}{\partial \xi^2} - (\alpha_1+\alpha_2)\,\frac{\partial^2 u}{\partial \xi \partial \eta} - \alpha_2\,\frac{\partial^2 u}{\partial \eta^2}\,. \end{aligned} \tag{III.2.9}$$

Inserting these relations into (III.2.6) yields the PDE in terms of the new coordinates,

$$\begin{aligned} \mathcal{L}u = {}& (A\,\alpha_1^2 - 2\,B\,\alpha_1 + C)\,\frac{\partial^2 u}{\partial \xi^2} + (A\,\alpha_2^2 - 2\,B\,\alpha_2 + C)\,\frac{\partial^2 u}{\partial \eta^2} \\ & +2\,(\alpha_1\,\alpha_2\,A - (\alpha_1+\alpha_2)\,B + C)\,\frac{\partial^2 u}{\partial \xi \partial \eta} \\ & +(-\alpha_1\,D + E)\,\frac{\partial u}{\partial \xi} + (-\alpha_2\,D + E)\,\frac{\partial u}{\partial \eta} + F\,u + G = 0\,. \end{aligned} \tag{III.2.10}$$

Let us choose the coefficients α to be the roots of

$$A\,\alpha^2 - 2\,B\,\alpha + C = 0, \tag{III.2.11}$$

namely,

$$\alpha_{1,2} = \frac{B}{A} \pm \frac{1}{A}\sqrt{B^2 - A\,C}\,. \tag{III.2.12}$$

Therefore we are led to distinguish three cases, depending on the nature of these roots. But before we enter this classification, we can make a very important observation:

the nature of the equation depends only on the coefficients of the second order terms. First order terms and zero order terms do not play a role here.

III.2.1.1 Hyperbolic equation $B^2 - A\,C > 0$, e.g. the wave equation

If the discriminant of the quadratic equation (III.2.11) is strictly positive, the two roots are real distinct, and the equation is said hyperbolic. The coefficient of the mixed second derivative of the equation does not vanish,

$$2\,(\alpha_1\,\alpha_2\,A - (\alpha_1+\alpha_2)\,B + C) = -\frac{4}{A}\sqrt{B^2 - A\,C} \neq 0\,. \tag{III.2.13}$$

The equation can then be cast in the canonical form,

$$\text{(H)}\quad \frac{\partial^2 u}{\partial\xi\partial\eta} + D'\,\frac{\partial u}{\partial\xi} + E'\,\frac{\partial u}{\partial\eta} + F'\,u + G' = 0\,, \tag{III.2.14}$$

where the superscript ' indicates that the original coefficients have been divided by the non zero term (III.2.13).

Another equivalent canonical form,

$$\text{(H)}\quad \frac{\partial^2 u}{\partial\sigma^2} - \frac{\partial^2 u}{\partial\tau^2} + D''\,\frac{\partial u}{\partial\sigma} + E''\,\frac{\partial u}{\partial\tau} + F''\,u + G'' = 0\,, \tag{III.2.15}$$

is obtained by the new set of coordinates,

$$\sigma = \frac{1}{2}\,(\xi+\eta), \quad \tau = \frac{1}{2}\,(\xi-\eta)\,. \tag{III.2.16}$$

The superscript $''$ in (III.2.15) indicates another modification of the original coefficients.

III.2.1.2 Parabolic equation $B^2 - A\,C = 0$, e.g. heat diffusion

A single family of characteristics exists, defined by

$$\alpha_1 = \alpha_2 = \frac{B}{A}\,. \tag{III.2.17}$$

A second arbitrary coordinate is introduced,

$$\xi = -\alpha\,x + y, \quad \eta = -\beta\,x + y, \quad \beta \neq \alpha\,, \tag{III.2.18}$$

which allows to cast the equation in the canonical form,

$$\text{(P)}\quad \frac{\partial^2 u}{\partial\eta^2} + D'\,\frac{\partial u}{\partial\xi} + E'\,\frac{\partial u}{\partial\eta} + F'\,u + G' = 0\,, \tag{III.2.19}$$

where the superscript ' indicates a modification of the original coefficients.

III.2.1.3 Elliptic equation $B^2 - AC < 0$, e.g. the laplacian

There are no real characteristics. Still, one may introduce the real coordinates,

$$\sigma = \frac{1}{2}(\xi + \eta) = -a\,x + y, \quad \tau = \frac{1}{2i}(\xi - \eta) = -b\,x\,, \tag{III.2.20}$$

with real coefficients a and b,

$$\alpha_{1,2} = a \pm i\,b, \quad a = \frac{B}{A}, \quad b = \frac{B}{A}\sqrt{AC - B^2}\,, \tag{III.2.21}$$

so as to cast the equation in the canonical form,

$$\text{(E)} \quad \frac{\partial^2 u}{\partial \sigma^2} + \frac{\partial^2 u}{\partial \tau^2} + D''\frac{\partial u}{\partial \sigma} + E''\frac{\partial u}{\partial \tau} + F''\,u + G'' = 0\,, \tag{III.2.22}$$

where the superscript ” indicates yet another modification of the original coefficients.

III.2.2 A single equation with variable coefficients

When the coefficients of the second order equation are variable, the analysis becomes more complex, but, fortunately, the main features of the constant case remain. Moreover, **the analysis below shows that this nature relies entirely on the sign of $B^2 - AC$, like in the constant coefficient equation**.

The characteristics are sought in the more general format,

$$\xi = \xi(x, y), \quad \eta = \eta(x, y)\,, \tag{III.2.23}$$

whence

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\,, \end{aligned} \tag{III.2.24}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial x^2}\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial x^2}\frac{\partial u}{\partial \eta} \\ \frac{\partial^2 u}{\partial y^2} &= \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial y^2}\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial y^2}\frac{\partial u}{\partial \eta} \\ \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\frac{\partial^2 u}{\partial \xi^2} + \left(\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}\right)\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial x \partial y}\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial x \partial y}\frac{\partial u}{\partial \eta}\,, \end{aligned} \tag{III.2.25}$$

yielding finally,

$$\mathcal{L}u = A'\frac{\partial^2 u}{\partial \xi^2} + 2\,B'\frac{\partial^2 u}{\partial \xi \partial \eta} + C'\frac{\partial^2 u}{\partial \eta^2} + D'\frac{\partial u}{\partial \xi} + E'\frac{\partial u}{\partial \eta} + F'\,u + G' = 0\,. \tag{III.2.26}$$

The coefficients of higher order,

$$A' = Q(\xi, \xi), \quad B' = Q(\xi, \eta), \quad C' = Q(\eta, \eta)\,, \tag{III.2.27}$$

are defined via the bilinear form Q,

$$Q(\xi,\eta) = A\,\frac{\partial \xi}{\partial x}\,\frac{\partial \eta}{\partial x} + B\,\Big(\frac{\partial \xi}{\partial x}\,\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\,\frac{\partial \eta}{\partial x}\Big) + C\,\frac{\partial \xi}{\partial y}\,\frac{\partial \eta}{\partial y}\,. \tag{III.2.28}$$

The remaining coefficients are,

$$\begin{aligned} D' &= D\,\frac{\partial \xi}{\partial x} + E\,\frac{\partial \xi}{\partial y} + A\,\frac{\partial^2 \xi}{\partial x^2} + 2\,B\,\frac{\partial^2 \xi}{\partial x \partial y} + C\,\frac{\partial^2 \xi}{\partial y^2} \\ E' &= D\,\frac{\partial \eta}{\partial x} + E\,\frac{\partial \eta}{\partial y} + A\,\frac{\partial^2 \eta}{\partial x^2} + 2\,B\,\frac{\partial^2 \eta}{\partial x \partial y} + C\,\frac{\partial^2 \eta}{\partial y^2} \\ F' &= F \\ G' &= G\,. \end{aligned} \tag{III.2.29}$$

Crucially,

$$B'^2 - A'C' = (B^2 - A\,C)\,\Big(\frac{\partial \xi}{\partial x}\,\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y}\,\frac{\partial \eta}{\partial x}\Big)^2\,. \tag{III.2.30}$$

Therefore, like for the constant coefficient equation, we are led to distinguish three cases, depending on the nature of the roots of a quadratic equation.

III.2.2.1 Hyperbolic equation $B^2 - A\,C > 0$, two real characteristics defined by $A' = C' = 0$, $B' \neq 0$

If $A = C = 0$, the original equation is already in the canonical format. Let us therefore consider the case $A \neq 0$.

The roots $f = \xi$ and η of $A' = 0$ and $C' = 0$ are,

$$\frac{\partial f/\partial x}{\partial f/\partial y} = a \pm b, \quad a = -\frac{B}{A}, \quad b = \frac{1}{A}\sqrt{B^2 - A\,C} \neq 0\,, \tag{III.2.31}$$

and, along the curves of slope

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -(a \pm b)\,, \tag{III.2.32}$$

f is constant,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0\,. \tag{III.2.33}$$

Consequently, these curves are the characteristics we were looking for.

III.2.2.2 Parabolic equation $B^2 - A\,C = 0$, one real characteristic defined by $A' = B' = 0$, $C' \neq 0$

A single family of characteristics exists, defined by $A' = 0$,

$$\frac{\partial \xi/\partial x}{\partial \xi/\partial y} = a = -\frac{B}{A}\,. \tag{III.2.34}$$

A second family of curves $\eta(x,y)$ is introduced, arbitrary but not parallel to the curves ξ constant,

$$\frac{\partial \eta/\partial x}{\partial \eta/\partial y} \neq \frac{\partial \xi/\partial x}{\partial \xi/\partial y} = a\,. \tag{III.2.35}$$

On the other hand, since $B/A = C/B = -a$, B' defined by eqns (III.2.27)-(III.2.28),

$$\begin{aligned} B' &= \underbrace{\left(A\,\frac{\partial\xi}{\partial x} + B\,\frac{\partial\xi}{\partial y}\right)}_{B/A=-a}\,\frac{\partial\eta}{\partial x} + \underbrace{\left(B\,\frac{\partial\xi}{\partial x} + C\,\frac{\partial\xi}{\partial y}\right)}_{C/B=-a}\,\frac{\partial\eta}{\partial y} \\ &= A\,\underbrace{\left(\frac{\partial\xi}{\partial x} - a\,\frac{\partial\xi}{\partial y}\right)}_{=0}\,\frac{\partial\eta}{\partial x} + \frac{AC}{B}\,\underbrace{\left(\frac{\partial\xi}{\partial x} - a\,\frac{\partial\xi}{\partial y}\right)}_{=0}\,\frac{\partial\eta}{\partial y}\,, \end{aligned} \tag{III.2.36}$$

vanishes, due to (III.2.34), but, as a consequence of the inequality (III.2.35),

$$C' = \left(\frac{\partial\eta}{\partial x} + \frac{C}{B}\,\frac{\partial\eta}{\partial y}\right)\left(A\,\frac{\partial\eta}{\partial x} + B\,\frac{\partial\eta}{\partial y}\right) \neq 0\,, \tag{III.2.37}$$

does not vanish.

III.2.2.3 Elliptic equation $B^2 - AC < 0$, two complex characteristics defined by $B' = 0$, $A' = C' \neq 0$

There are no real characteristics. The roots of $Q(f,f) = 0$ are complex,

$$\frac{\partial\xi/\partial x}{\partial\xi/\partial y} = a + i\,b, \quad a = \frac{B}{A}, \quad b = \frac{B}{A}\sqrt{AC - B^2}\,. \tag{III.2.38}$$

Still, one may introduce the real coordinates,

$$\sigma = \frac{1}{2}\,(\xi + \eta), \quad \tau = \frac{1}{2i}\,(\xi - \eta)\,. \tag{III.2.39}$$

Inserting

$$\xi = \sigma + i\,\tau, \quad \eta = \sigma - i\,\tau\,, \tag{III.2.40}$$

into B', eqns (III.2.27)-(III.2.28), yields,

$$Q(\sigma,\sigma) - Q(\tau,\tau) + 2\,i\,Q(\sigma,\tau) = 0\,, \tag{III.2.41}$$

and therefore,

$$A' = Q(\sigma,\sigma) = Q(\tau,\tau) = C' \neq 0, \quad B' = Q(\sigma,\tau) = 0\,. \tag{III.2.42}$$

The reals $A' = C'$ do not vanish because the roots ξ and η of $Q(f,f) = 0$ are complex.

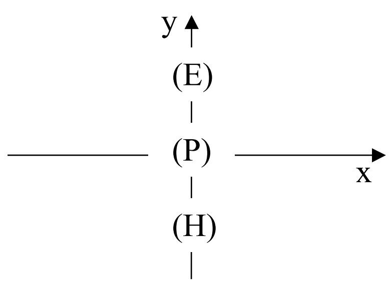

Figure III.2 Tricomi equation of transonic flow provides a conspicuous example of second order equation with variable coefficients where the type varies pointwise.

Remark 1: the Tricomi equation of fluid dynamics

The type of a nonlinear equation may change pointwise. The prototype that illustrates best this issue is the Tricomi equation of transonic flow,

$$\frac{\partial^2 u}{\partial x^2} + y\,\frac{\partial^2 u}{\partial y^2} = 0\,, \tag{III.2.43}$$

corresponding to $A = 1$, $B = 0$, $C = y$, so that the nature of the equation depends on $B^2 - A\,C = -y$, whence the types displayed on Fig. III.2.

At this point, we should emit a warning. Even if, in this equation, the boundary in the plane (x, y) between the (H) and (E) types is of (P) type, this is by no means a general situation.

Remark 2: are the classifications of first and second order equations compatible?

Note that we have used the same terminology to class the types of equations, whether first order or second order. This was perhaps a bit too presumptuous. Indeed, for example, a second order equation can be written in the format of two first order equations, and conversely. Examples are provided in Exercises III.1, and III.6. Therefore, we have defined two classifications for second order equations, that of Sect. III.1, and that associated to the set of two first order equations exposed in Sect. III.2.

As they say in French, we looked for the stick to be beaten. However, no worry, man, we are safe! This is because the classifications were built on physical grounds, that is, on the interpretations exposed at length in the previous chapters. Whether written in one form or the other, equations convey the same physical information.

As an illustration, a set of two first order hyperbolic equations is considered in Exercise III.1. The associated second order equation turns out to propagate disturbances at the same speed as the first order set, and it is therefore hyperbolic as well!

III.2.3 Properties of real characteristics

III.2.3.1 The equation of the characteristics

In the previous section, we have shown that the existence of real characteristics corresponds to either $A' = 0$, or $C' = 0$, or both. Let $f = f(x, y) =$ constant be the analytical expression of such a real characteristic. Thus, along a characteristic,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0\,, \tag{III.2.44}$$

and, therefore,

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}\,. \tag{III.2.45}$$

Inserting (III.2.45) into $Q(f, f)$ defined by (III.2.28) yields the equation that provides the slope(s) of the real characteristic(s),

$$A\,(dy)^2 - 2\,B\,dy\,dx + C\,(dx)^2 = 0\,. \tag{III.2.46}$$

Please pay attention to the sign in front of the mixed term.

III.2.3.2 Indeterminacy of the Cauchy problem

The characteristics may be given another definition:

these are the curves along which the Cauchy problem is indeterminate or impossible

The issue is the following:

- consider a function u that satisfies the equation (III.2.6);
- prescribe u, $\partial u/\partial x$, and $\partial u/\partial y$;
- obtain the three second order derivatives of u in terms of u, $\partial u/\partial x$, and $\partial u/\partial y$.

The 3×3 linear system to be solved is,

$$\begin{bmatrix} A & 2B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} \partial^2 u/\partial x^2 \\ \partial^2 u/\partial x \partial y \\ \partial^2 u/\partial y^2 \end{bmatrix} = \begin{bmatrix} -D\,\partial u/\partial x - E\,\partial u/\partial y - F\,u - G \\ d(\partial u/\partial x) \\ d(\partial u/\partial y) \end{bmatrix}. \qquad \text{(III.2.47)}$$

That the matrix displayed here is singular along the characteristics curves defined by (III.2.46) is easily checked.

Another way to express the indeterminacy of the Cauchy problem is to state that characteristics are the sole curves along which discontinuities can be propagated.

III.3 Extension to more than two variables

The classification can be extended to equations of order higher than 2, and depending on more than 2 variables. For example, let us consider the second order equation depending on n variables,

$$\sum_{i,j=1}^{n} a_{ij}\,\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_i\,\frac{\partial u}{\partial x_i} + c\,u + g = 0\,. \qquad \text{(III.3.1)}$$

The coefficient matrix in (III.3.1) should be symmetrized because we have tacitly assumed the partial derivatives to commute, namely $\partial^2/\partial x_i \partial x_j = \partial^2/\partial x_j \partial x_i$, for any i and j in $[1, n]$.

The classification is as follows:

- (H) for ($Z = 0$ and $P = 1$) or ($Z = 0$ and $P = n - 1$)
- (P) for $Z > 0$ ($\Leftrightarrow \det \mathbf{a} = 0$)
- (E) for ($Z = 0$ and $P = n$) or ($Z = 0$ and $P = 0$)
- (ultraH) for ($Z = 0$ and $1 < P < n - 1$)

where

- Z: nb. of zero eigenvalues of $\mathbf{a}$
- P: nb. of strictly positive eigenvalues of $\mathbf{a}$

The alternatives in the (H) and (P) definitions are due to the fact that multiplication by -1 of the equation leaves it unchanged.

The canonical forms in the characteristic coordinates $\xi' s$ generalize the previous expressions for two variables:

$$
\begin{aligned}
&(\mathrm{H}) \quad \frac{\partial^2 u}{\partial \xi_1^2} - \sum_{i=2}^{n} \frac{\partial^2 u}{\partial \xi_i^2} \\
&(\mathrm{P}) \quad \sum_{i=1}^{n-Z} \pm \frac{\partial^2 u}{\partial \xi_i^2} \\
&(\mathrm{E}) \quad \sum_{i=1}^{n} \frac{\partial^2 u}{\partial \xi_i^2} \\
&(\mathrm{uH}) \quad \sum_{i=1}^{P} \frac{\partial^2 u}{\partial \xi_i^2} - \sum_{i=P+1}^{n} \frac{\partial^2 u}{\partial \xi_i^2}
\end{aligned}
\tag{III.3.2}
$$

To make the link with the analysis of the previous section, set

$$
n=2, \qquad \begin{bmatrix} A=a_{11} & B=a_{12} \\ B=a_{21} & C=a_{22} \end{bmatrix}, \tag{III.3.3}
$$

whence,

$$
\begin{aligned}
&\det(\mathbf{a} - \lambda\, \mathbf{I}) = \lambda^2 - (A+C)\,\lambda + A\,C - B^2 = 0 \\
&\Leftrightarrow \begin{cases} \lambda_1\,\lambda_2 < 0 & \Leftrightarrow \quad A\,C - B^2 < 0 \quad (\mathrm{H}) \\ \lambda_1\,\lambda_2 > 0 & \Leftrightarrow \quad A\,C - B^2 > 0 \quad (\mathrm{E}) \\ \lambda = 0 & \Leftrightarrow \quad A\,C - B^2 = 0 \quad (\mathrm{P}) \end{cases}
\end{aligned}
\tag{III.3.4}
$$

Example: consider the second order equation for the unknown $u(x,y,z)$,

$$
3\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 4\,\frac{\partial^2 u}{\partial y \partial z} + 4\,\frac{\partial^2 u}{\partial z^2} = 0\,. \tag{III.3.5}
$$

Its nature is obtained by inspecting the spectral properties of the symmetric matrix

$$
\mathbf{a} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \quad \Rightarrow \quad \det(\mathbf{a} - \lambda\,\mathbf{I}) = (3-\lambda)\,\lambda\,(\lambda - 5)\,, \tag{III.3.6}
$$

which turns out to have a zero eigenvalue so that the equation is parabolic.

Exercise III.1: **1D-waves in shallow water**.

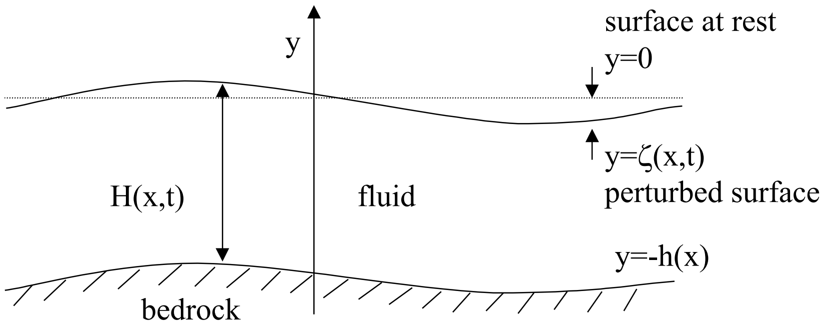

Figure III.3 In shallow water channels, horizontal wavelengths are longer than the depth.

Disturbances at the surface of a fluid surface may give rise to waves because gravity tends to restore equilibrium. In a shallow channel, filled with an incompressible fluid with low viscosity, horizontal wavelengths are much larger than the depth, and water flows essentially in the horizontal directions. In this circumstance, the equations that govern the motion of the fluid take a simplified form. To simplify further the problem, the horizontal flow is restricted to one direction along the x-axis.

Let $u(x,t)$ be the horizontal particle velocity, $\zeta(x,t)$ the position of the perturbed free surface, and $h(x)$ the vertical position of the fixed bedrock. Then

$$H(x,t) = h(x) + \zeta(x,t)\,, \tag{1}$$

is the water height. Mass conservation,

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (uH)}{\partial x} = 0\,, \tag{2}$$

and horizontal balance of momentum, involving the gravitational acceleration g,

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + g\,\frac{\partial \zeta}{\partial x} = 0, \tag{3}$$

are the two coupled nonlinear equations governing the unknown velocity $u(x,t)$ and fluid height $H(x,t)$.

If we were interested in solving completely the associated IBVP, we should prescribe boundary conditions and initial conditions. However, here, we are only interested in checking the nature of the field equations (FE).

1. In the case of an horizontal bedrock $h(x) = h$ constant, show that the system of equations remains coupled, and find its nature.
2. Give an interpretation to your finding. Hint: linearize the equations.
3. Define the Riemann invariants.
4. Show that the set of the two equations is a conservation law, in the sense of (III.1.20).

Solution:

The system of equations is first cast into the standard format (III.1.10),

$$\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} H \\ u \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} u & H \\ g & u \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} H \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{4}$$

The resulting eigenvalue problem (III.1.12),

$$\boldsymbol{\lambda} \cdot (\mathbf{a}\frac{dx}{dt} - \mathbf{b}) = \mathbf{0}\,, \tag{5}$$

yields two real distinct eigenvalues, and associated independent eigenvectors,

$$\frac{dx_+}{dt} = u + \sqrt{gH}, \quad \boldsymbol{\lambda}_+ = \begin{bmatrix} g \\ \sqrt{gH} \end{bmatrix}; \quad \frac{dx_-}{dt} = u - \sqrt{gH}, \quad \boldsymbol{\lambda}_- = \begin{bmatrix} g \\ -\sqrt{gH} \end{bmatrix}, \tag{6}$$

so that the system is hyperbolic, that is, it is expected to be able to propagate disturbances at finite speed.

2. Can we define this speed? To clarify this issue, let us linearize the equations around $u = 0$, $\zeta = 0$,

$$\begin{aligned} \frac{\partial \zeta}{\partial t} + H\,\frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} + g\,\frac{\partial \zeta}{\partial x} &= 0\,. \end{aligned} \tag{7}$$

Applying the operator $-g\,\partial/\partial x$ to the first line, and $\partial/\partial t$ to the second line, and adding the results yields the second order wave equation,

$$\frac{\partial^2 u}{\partial t^2} - (\sqrt{gH})^2 \frac{\partial^2 u}{\partial x^2} = 0\,, \tag{8}$$

which shows that

$$c = \sqrt{gH}\,, \tag{9}$$

is the wave-speed at which infinitesimal second order disturbances propagate.

Well, that is fine, but we are not totally satisfied because we started from first order equations. Indeed, let us seek if first order waves of the form,

$$\begin{aligned} u(x,t) &= u_+(x + c\,t) + u_-(x - c\,t) \\ \zeta(x,t) &= \zeta_+(x + c\,t) + \zeta_-(x - c\,t)\,, \end{aligned} \tag{10}$$

can propagate to the right and to the left at the very same speed c, as second order waves. Inserting the expressions (10) in (7) shows that indeed these waves can propagate for arbitrary u_+ and u_- and for

$$\zeta_+(x + c\,t) = -\frac{c}{g}\,u_+(x + c\,t), \quad \zeta_-(x - c\,t) = \frac{c}{g}\,u_-(x - c\,t)\,. \tag{11}$$

3. We now come back to the nonlinear system. For each characteristic, the Riemann invariants are defined via (III.1.17), which specializes here to,

$$\boldsymbol{\lambda} \cdot \frac{d\mathbf{u}}{d\sigma} = 0\,. \tag{12}$$

Substituting c for $H = c^2/g$ yields

$$\frac{d}{d\sigma_\pm}(u \pm 2\,c) = 0 \quad \text{along the characteristic} \quad \frac{dx}{dt} = u \pm c\,. \tag{13}$$

The interpretation is as follows: $u + 2c$ is constant along the characteristic $dx/dt = u + c$, and $u - 2c$ is constant along the characteristic $dx/dt = u - c$.

4. Indeed, the system can be recast into the format (III.1.20),

$$\frac{\partial}{\partial t}\overbrace{\begin{bmatrix} H \\ u \end{bmatrix}}^{\mathbf{u}} + \frac{\partial}{\partial x}\overbrace{\begin{bmatrix} u\,H \\ g\,H + u^2/2 \end{bmatrix}}^{\mathbf{G}(\mathbf{u})} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{14}$$

For those who want to know more.

Mass conservation is obtained by considering a vertical column of height H and constant horizontal area S, and requiring the time rate of change of its mass $M = \rho S H$ to be equal to the flux $M\mathbf{v}$ traversing the column,

$$\frac{\partial M}{\partial t} + \operatorname{div}(M\mathbf{v}) = 0\,. \tag{15}$$

Since the density ρ is constant, mass conservation,

$$\frac{\partial \zeta}{\partial t} + \operatorname{div}(H\mathbf{v}) = 0\,, \tag{16}$$

simplifies to equation (2) since the particle velocity $\mathbf{v}$ is essentially horizontal.

Momentum balance expresses in terms of the gradient of pressure ∇p, vertical gravitational acceleration $\mathbf{g}$, and particle acceleration $d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + \mathbf{v}\cdot\nabla\mathbf{v}$,

$$-\nabla p + \rho\left(\mathbf{g} - \frac{\partial \mathbf{v}}{\partial t} - \mathbf{v}\cdot\nabla\mathbf{v}\right) = \mathbf{0}\,. \tag{17}$$

For shallow waters, the vertical component of the momentum balance is dominated by the pressure gradient and gravity terms, yielding the hydrostatic pressure $p = \rho\,g\,(\zeta - y)$. Inserting this expression in the horizontal component of the momentum balance yields equation (3), once again under the assumption of an essentially horizontal flow.

Exercise III.2: **Switching from first and to second order equations, and conversely**.

1. Define the nature of the set of first order equations,

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0\,, \quad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0\,. \tag{1}$$

Obtain the equivalent second order equation. Was the nature of the system unexpected?

2. Consider the heat equation,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0\,, \tag{2}$$

which is the prototype of a parabolic equation. Obtain an equivalent set of two first order equations. Analyze its nature.

3. Consider the telegraph equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + a\,\frac{\partial u}{\partial t} + b\,u = 0\,, \tag{3}$$

where a, b and c are positive quantities, possibly dependent on position. Define the nature of this equation. Obtain an equivalent first order system of partial differential equations.

Solution:

1. Of course, we recognize the Cauchy Riemann equations. The system of equations can be cast into the standard format (III.1.10),

$$\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial x} \overbrace{\begin{bmatrix} u \\ v \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial y} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4}$$

The resulting eigenvalue problem (III.1.12) implies $\det(\mathbf{a}\,\lambda - \mathbf{b}) = \lambda^2 + 1 = 0$ with $\lambda = dy/dx$, so that the eigenvalues are complex, and the system is therefore elliptic, according to the terminology of Sect. III.1.2.

Now, a basic manipulation of the equations,

$$\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\Big) + \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) = \Delta u = 0\,, \tag{5}$$

shows, assuming sufficient smoothness, that u is harmonic, and therefore solution of an elliptic second order equation. So is v, namely $\Delta v = 0$, due to the fact that, if the set $((x, y), (u, v))$ satisfies the Cauchy Riemann equations, so does the set $((y, x), (v, u))$.

2. For example, we may set $v = \partial u/\partial x$. The first order equivalent system becomes,

$$\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} u \\ v \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ -v \end{bmatrix}}^{\mathbf{c}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

The resulting eigenvalue problem, with $\lambda = dx/dt$, implies $\det(\mathbf{a}\,\lambda - \mathbf{b}) = 1 \cdots$ strange $\cdots$ Never mind! We should not be deterred at the first difficulty. Let us change the angle of attack, and consider the eigenvalue problem, $\boldsymbol{\lambda} \cdot (\mathbf{a} - \mathbf{b}\,\lambda) = \mathbf{0}$, with associated characteristic polynomial,

$$\det\,(\mathbf{a} - \mathbf{b}\,\lambda) = \lambda^2\,, \tag{7}$$

where now $\lambda = dt/dx$. Therefore, $\lambda = 0$ is an eigenvalue of algebraic multiplicity 2. The associated eigenspace, generated by the vectors $\boldsymbol{\lambda}$ such that,

$$\boldsymbol{\lambda} \cdot (\mathbf{a} - \mathbf{b}\,\lambda) = \boldsymbol{\lambda} \cdot \mathbf{a} = \mathbf{0}\,, \tag{8}$$

is in fact spanned by the sole eigenvector $\boldsymbol{\lambda} = [0, 1]$. Therefore, the generalized eigenvalue problem (8) is defective, and the set of the two first order equations is parabolic, according to the terminology of Sect. III.1.2. Thus we have another example where the terminologies used to class first and second order equations are consistent.

3. The telegraph equation is clearly hyperbolic, and $c > 0$ is the wave speed.

With $u_1 = u$, $u_2 = \partial u/\partial x$ and $u_3 = \partial u/\partial t$, the telegraph equation may be equivalently expressed as a first order system of PDEs,

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -c^2 & 0 \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \overbrace{\begin{bmatrix} -u_3 \\ 0 \\ c^2\,(a\,u_3 + b\,u_1) \end{bmatrix}}^{\mathbf{c}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

The generalized eigenvalues $dx/dt = 0, \pm\, c$ are real and distinct.

Exercise III.3: **Air compressibility in high-speed aerodynamics**.

Air compressibility can not be neglected in high-speed aerodynamics. If air is assumed to be a perfect gas, its pressure p and density ρ are linked by the constitutive relation $p/p_0 = (\rho/\rho_0)^\gamma$, (p_0, ρ_0) being reference values and $\gamma > 0$ is a material constant, equal to the ratio of heat capacities. It is instrumental to define a quantity c, that can be interpreted as a wave-speed,

$$c^2 = \frac{dp}{d\rho} = \gamma\,\frac{p_0}{\rho_0^\gamma}\,\rho^{\gamma-1} = \gamma\,\frac{p}{\rho}\,, \tag{1}$$

and therefore,

$$dp = c^2\,d\rho, \quad d\rho = \frac{2}{\gamma-1}\,\frac{\rho}{c}\,dc\,. \tag{2}$$

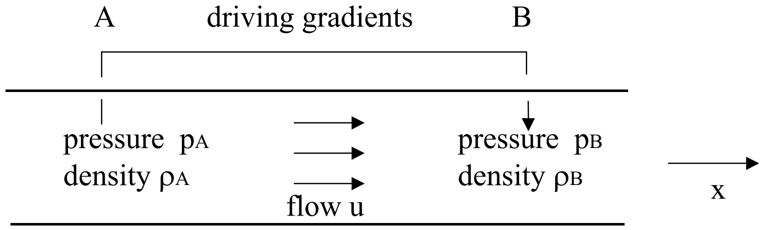

Figure III.4 The flow of ideal gas in a tube is triggered by differences of pressure and density at the ends of the tube.

The one-dimensional flow u of an ideal gas in a tube of axis x is triggered by gradients of pressure $p > 0$ and density $\rho > 0$. The field equations governing these three unknown functions of space x and time t are the three coupled nonlinear partial differential equations,

$$\begin{aligned}
\text{mass conservation :}\quad & \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0\\
\text{momentum conservation :}\quad & \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + \frac{1}{\rho}\,\frac{\partial p}{\partial x} = 0\\
\text{constitutive equation :}\quad & \frac{\partial p}{\partial t} + u\,\frac{\partial p}{\partial x} + \rho\,c^2\,\frac{\partial u}{\partial x} = 0\,.
\end{aligned} \tag{3}$$

1. Define the nature of this set of first order equations, the eigenvalues and eigenvectors associated to the characteristic problem, and the Riemann invariants along each characteristic.

2. Show that this system is of divergence type along the definition (III.1.19).

Solution:

1. The system of equations can be cast into the standard format (III.1.10),

$$\begin{array}{c} (a)\\ (b)\\ (c)\end{array}\quad \overbrace{\begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}}^{\mathbf{a}}\,\frac{\partial}{\partial t}\overbrace{\begin{bmatrix}\rho\\ u\\ p\end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} u & \rho & 0\\ 0 & u & 1/\rho\\ 0 & \rho\,c^2 & u\end{bmatrix}}^{\mathbf{b}}\,\frac{\partial}{\partial x}\begin{bmatrix}\rho\\ u\\ p\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}. \tag{4}$$

The characteristic equation,

$$\det(\mathbf{a}\,\lambda - \mathbf{b}) = (\lambda - u)\,\Big((\lambda - u)^2 - c^2\Big) = 0\,, \tag{5}$$

provides three real distinct eigenvalues and independent (left) eigenvectors,

$$\lambda_1 = u\,, \quad \boldsymbol{\lambda}_1 = \begin{cases} 1 \\ 0 \\ 0 \end{cases} ; \quad \lambda_{2,3} = u \pm c\,, \quad \boldsymbol{\lambda}_{2,3} = \begin{cases} 0 \\ 1 \\ \pm 1/(\rho c) \end{cases} , \tag{6}$$

so that the system is hyperbolic.

The Riemann invariants along each characteristic are obtained by (III.1.17), namely, $d\rho = 0$ along the first characteristic, and, along the second and third characteristics,

$$d\Big(u \pm \frac{2}{\gamma - 1}\,c\Big) = 0\,. \tag{7}$$

2. Indeed, the system can be cast in the format,

$$\begin{array}{l} (a)' \\ (b)' \\ (c)' \end{array} \quad \frac{\partial}{\partial t}\overbrace{\begin{bmatrix} \rho \\ \rho u \\ p + (\gamma - 1)\,\dfrac{\rho u^2}{2} \end{bmatrix}}^{\mathbf{F}(\mathbf{u})} + \frac{\partial}{\partial x}\overbrace{\begin{bmatrix} \rho u \\ p + \rho u^2 \\ \Big(p + (\gamma - 1)\,(p + \dfrac{\rho u^2}{2})\Big)\,u \end{bmatrix}}^{\mathbf{G}(\mathbf{u})} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{8}$$

The proof is as follows:
2.1 (a)'=(a).
2.2 (b)'=$\rho\times$(b)+$u\times$(a).
2.3

$$\begin{aligned} (c)' &= \frac{\partial p}{\partial t} + \frac{\gamma - 1}{2}\,\frac{\partial}{\partial x}(\rho\,u^2) \\ &= \frac{\partial p}{\partial t} + \frac{\gamma - 1}{2}\,u^2\,\frac{\partial \rho}{\partial x} + (\gamma - 1)\,\rho\,u\,\frac{\partial u}{\partial x} \\ &= \overbrace{-u\,\frac{\partial p}{\partial x} - \gamma p \frac{\partial u}{\partial x}}^{(c)} - \overbrace{\frac{\gamma - 1}{2}\,u^2\,\frac{\partial (\rho u)}{\partial x}}^{(a)} - \overbrace{(\gamma - 1)\,\rho\,u\,(u\,\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x})}^{(b)} \\ &= -\gamma \frac{\partial (up)}{\partial x} - (\gamma - 1)\,\frac{\partial}{\partial x}(\rho\,\frac{u^3}{2}) \end{aligned} \tag{9}$$

$\square$.

Exercise III.4: **Second order equations**.

1. Define the nature of the second order equation,

$$3\,\frac{\partial^2 u}{\partial x^2} + 2\,\frac{\partial^2 u}{\partial x \partial y} + 5\,\frac{\partial^2 u}{\partial y^2} x\, + 2\,\frac{\partial u}{\partial y} = 0\,. \tag{1}$$

2. Consider the wave equation $c^2\,\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 = 0$, the heat equation $\partial u/\partial t - \partial^2 u/\partial x^2 = 0$, and the Laplacian $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$. Define the respective characteristic curves.

3. Indicate the nature, define the characteristics and cast into canonical form the second order equation,

$$e^{2x}\,\frac{\partial^2 u}{\partial x^2} + 2\,e^{x+y}\,\frac{\partial^2 u}{\partial x \partial y} + e^{2y}\,\frac{\partial^2 u}{\partial y^2} = 0\,. \tag{2}$$

4. Same questions for the second order equation,

$$2\,\frac{\partial^2 u}{\partial x^2} - 4\,\frac{\partial^2 u}{\partial x \partial y} - 6\,\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0\,. \tag{3}$$

Solution:

1.1 With the method developed in Sect. III.2, we identify $A = 3$, $B = 1$ $C = 5$, so that $B^2 - A\,C = -14 < 0$, and therefore the equation is elliptic.

1.2 With the method exposed in Sect. III.3, the symmetric matrix $\mathbf{a}$ is identified as,

$$\mathbf{a} = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}. \tag{4}$$

The characteristic equation becomes $\det(\mathbf{a} - \lambda\,\mathbf{I}) = \lambda^2 - 8\,\lambda + 14 = 0$, whose roots are real, positive and distinct, so that the conclusion above is retrieved!

2. Equation (III.2.46) gives the slope of the real characteristics of a second order equation. Thus, the characteristics are, respectively for the wave equation,

$$\left.\begin{array}{l} c^2\,\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = 0 \\ c^2\,(dt)^2 - \,(dx)^2 = 0 \end{array}\right\} \quad x \pm a\,t = \text{constant}; \tag{5}$$

for the heat equation,

$$\left.\begin{array}{l} D\,\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial u}{\partial t} = 0 \\ D\,(dt)^2 = 0 \end{array}\right\} \quad t = \text{constant}; \tag{6}$$

for the Laplacian,

$$\left.\begin{array}{l} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 \\ (dx)^2 + \,(dy)^2 = 0 \end{array}\right\} \quad \text{no real characteristics}\,. \tag{7}$$

3. Along (III.2.46), the slope(s) of the characteristic(s) is(are) defined by the equation,

$$e^{2x}\,(dy)^2 - 2\,e^{x+y}\,dy\,dx + e^{2y}\,(dx)^2 = (e^x\,dy - e^y\,dx)^2 = 0\,, \tag{8}$$

and therefore there is a single characteristic $\xi(x,y) = e^{-x}\,dx - e^{-y}\,dy =$ constant, and the equation is parabolic. A second arbitrary coordinate may be defined, e.g. $\eta(x,y) = x \neq \xi(x,y)$. Using the relations (III.2.25) between partial derivatives, the equation becomes

$$e^{2\,\eta}\,\frac{\partial^2 u}{\partial \xi^2} + \cdots \frac{\partial u}{\partial \eta} = 0\,, \tag{9}$$

in terms of the coordinates (ξ, η).

4. Along (III.2.46), the slope(s) of the characteristic(s) is(are) defined by the equation,

$$2\,(dy)^2 + 4\,dy\,dx - 6\,(dx)^2 = 2\,(dy + 3\,dx)\,(dy - dx) = 0\,, \tag{10}$$

and therefore there are two characteristic $\xi(x,y) = -x + y$ constant, $\eta(x,y) = 3\,x + y$ constant, and the equation is hyperbolic. Using the relations (III.2.25) between partial derivatives, the equation becomes

$$-32\,\frac{\partial^2 u}{\partial \xi \partial \eta} - \frac{\partial u}{\partial \xi} + 3\,\frac{\partial u}{\partial \eta} = 0\,, \tag{11}$$

in terms of the coordinates (ξ, η).

Exercise III.5: **Normal form of a hyperbolic system**.

The unknown vector $\mathbf{u} = \mathbf{u}(x,t)$, of size n, obeys the first order differential system,

$$\mathbf{a} \cdot \frac{\partial \mathbf{u}}{\partial t} + \mathbf{b} \cdot \frac{\partial \mathbf{u}}{\partial x} + \mathbf{c} = \mathbf{0}, \tag{1}$$

where $\mathbf{a}$, $\mathbf{b}$ are non singular constant matrices and $\mathbf{c}$ is a vector.

1-a The system is said under *normal form* if the matrix $\mathbf{a}$ can be decomposed into a product of a diagonal matrix $\mathbf{d}$ times the matrix $\mathbf{b}$,

$$\mathbf{a} = \mathbf{d} \cdot \mathbf{b}\,. \tag{2}$$

Show that a normal system is hyperbolic.

1-b Conversely, if one admits that, for any non singular matrices $\mathbf{a}$ and $\mathbf{b}$, there exist a non singular matrix $\mathbf{t}$ and a non singular diagonal matrix $\mathbf{d}$, such that,

$$\mathbf{t} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{t} \cdot \mathbf{b}\,, \tag{3}$$

show that any hyperbolic system can be written in normal form.

2-a Consider now the particular matrices and vectors,

$$\mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 2 & -2 \\ 1 & -4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -v \\ -u \end{bmatrix}. \tag{4}$$

Show that the system is hyperbolic, define the characteristics, and write it in normal form.

2-b Consider now the above homogeneous system, i.e. with $\mathbf{c} = \mathbf{0}$. Given initial data, namely $u(x, t = 0) = u_0(x)$, $v(x, t = 0) = v_0(x)$, solve the system of partial differential equations.

Solution:

1-a. The nature of the system is defined by the spectral properties of the pencil $(\mathbf{a}, \mathbf{b})$, with characteristic polynomial,

$$\det\,(\mathbf{a} - \mathbf{b}\frac{dt}{dx}) = \det\,(\mathbf{d} - \mathbf{I}\frac{dt}{dx}) \det\,\mathrm{b} = 0\,, \tag{5}$$

so that the eigenspace generates $\mathbb{R}^n$, and the i-th vector is associated to the eigenvalue $(dt/dx)_i = d_i$, $i \in [1, n]$. Note that since $\mathbf{a}$ and $\mathbf{b}$ are non singular, so is $\mathbf{d}$.

1-b. Pre-multiplication of (6) by $\mathbf{t}$ yields,

$$\begin{array}{rcccccl} \mathbf{t} \cdot \mathbf{a} \cdot \dfrac{\partial \mathbf{u}}{\partial t} & + & \mathbf{t} \cdot \mathbf{b} \cdot \dfrac{\partial \mathbf{u}}{\partial x} & + & \mathbf{t} \cdot \mathbf{c} & = & \mathbf{0} \\ \mathbf{d} \cdot \tilde{\mathbf{b}} \cdot \dfrac{\partial \mathbf{u}}{\partial t} & + & \tilde{\mathbf{b}} \cdot \dfrac{\partial \mathbf{u}}{\partial x} & + & \tilde{\mathbf{c}} & = & \mathbf{0}, \end{array} \tag{6}$$

with $\tilde{\mathbf{b}} = \mathbf{t} \cdot \mathbf{b}$, $\tilde{\mathbf{c}} = \mathbf{t} \cdot \mathbf{c}$.

2-a. The eigenvalues,

$$\det\,(\mathbf{a} - \mathbf{b}\frac{dt}{dx}) = -\,(2 + \frac{dt}{dx})\,(3 - \frac{dt}{dx}) = 0\,, \tag{7}$$

are real and distinct. The characteristics are the lines,

$$\xi = t + 2\,x \text{ const}, \quad \eta = t - 3\,x \text{ const}\,. \tag{8}$$

Let $\mathbf{d} = \mathbf{diag}[-2, 3]$ be the diagonal matrix of the eigenvalues. Then the matrix $\mathbf{t}$ has to satisfy the equations,

$$\mathbf{a}\cdot\mathbf{a} = \mathbf{d}\cdot\mathbf{d}\cdot\mathbf{b} \quad \Leftrightarrow \quad \begin{bmatrix} 2\,t_{11} + t_{12} & -2\,t_{11} - 4\,t_{12} \\ 2\,t_{21} + t_{22} & -2\,t_{21} - 4\,t_{22} \end{bmatrix} = \begin{bmatrix} -2\,t_{11} & 6\,t_{11} - 2\,t_{12} \\ 3\,t_{21} & -9\,t_{21} + 3\,t_{22} \end{bmatrix}. \tag{9}$$

The matrix $\mathbf{t}$ can be defined to within two arbitrary degrees of freedom, namely $t_{12} = -4\,t_{11}$, $t_{21} = t_{22}$. One may take,

$$\mathbf{t} = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}, \tag{10}$$

and the system (6) then writes,

$$\begin{bmatrix} -2 & 14 \\ 3 & -6 \end{bmatrix} \cdot \frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 & -7 \\ 1 & -2 \end{bmatrix} \cdot \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 4\,u - v \\ -u - v \end{bmatrix} = \mathbf{0}, \tag{11}$$

or equivalently,

$$\begin{aligned} \Big(-2\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big) \quad (u - 7\,v) \quad +4\,u - v &= 0 \\ \Big(3\,\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\Big) \quad (u - 2\,v) \quad -u - v &= 0\,, \end{aligned} \tag{12}$$

or in terms of the coordinates (ξ, η),

$$\begin{aligned} -5\,\frac{\partial}{\partial \eta}\,(u - 7\,v) \quad +4\,u - v &= 0 \\ 5\,\frac{\partial}{\partial \xi}\,(u - 2\,v) \quad -u - v &= 0\,. \end{aligned} \tag{13}$$

2-b. The homogeneous system,

$$\frac{\partial}{\partial \eta}\,(u - 7\,v) = 0, \quad \frac{\partial}{\partial \xi}\,(u - 2\,v) = 0\,, \tag{14}$$

displays the Riemann invariants in explicit form,

$$\begin{aligned} u - 7\,v &= (u - 7\,v)(E_0) \text{ constant along the characteristic } \eta = \text{const}\,, \\ u - 2\,v &= (u - 2\,v)(X_0) \text{ constant along the characteristic } \xi = \text{const}\,. \end{aligned} \tag{15}$$

Let $P(x,t)$ an arbitrary point, and $X_0(x + t/2, 0)$, $E_0(x - t/3, 0)$ the points of the x-axis from which the characteristics that meet at point P emanates. The solution at an arbitrary point (x,t) reads,

$$\begin{aligned} u(x,t) &= \frac{7}{5}(u_0 - 2\,v_0)(X_0) - \frac{2}{5}(u_0 - 7\,v_0)(E_0) \\ v(x,t) &= \frac{1}{5}(u_0 - 2\,v_0)(X_0) - \frac{1}{5}(u_0 - 7\,v_0)(E_0)\,. \end{aligned} \tag{16}$$

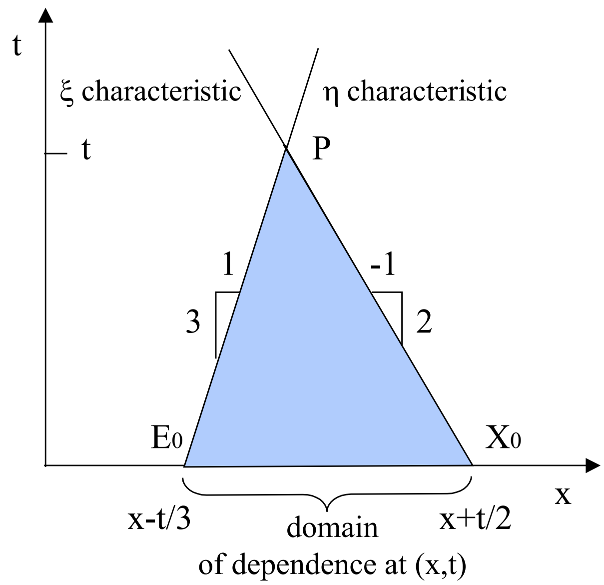

Figure III.5 Given initial data on the x-axis, the solution is built from the characteristic network, playing with the Riemann invariants. The region $\mathrm{E}_0\mathrm{PX}_0$ is referred to as the **domain of dependence** of the solution at point $\mathrm{P}(x,t)$: indeed, change of the initial conditions outside the interval $\mathrm{E}_0\mathrm{X}_0$ will not modify this solution.

Exercise III.6: **Transmission lines and the telegraph equation**.

An electrical circuit representing a transmission line is shown on Fig. III.6. It involves an inductance $L = L(x)$; a resistance $R = R(x)$; a capacitance to ground $C = C(x)$; a conductance to ground $G = G(x)$. In a first step, all these material properties are assumed to be strictly positive. They may vary along the line.

The variation of the potential dV over a segment of length dx is due to the resistance $R\,dx\,I$ and to the inductance $L\,dx\,dI/dt$. Let $q = C\,dx\,V$ be the charge across the capacitor. The variation of the current dI is due to the capacitance $C\,dx\,dV/dt$ and to the conductance $G\,dx\,V$.

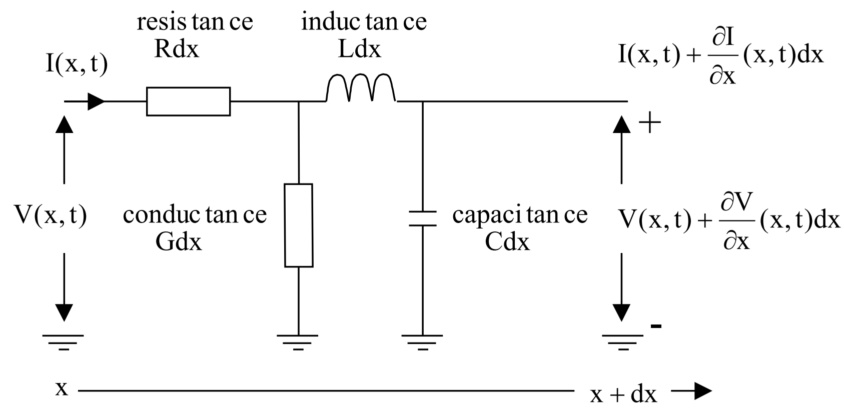

Figure III.6 Elementary circuit of length dx used as a model of the transmission lines.

1-a Therefore, the equations governing the current $I(x,t)$ and potential $V(x,t)$ in a transmission line of axis x can be cast in the format of a linear system of two partial differential equations,

$$\overbrace{\begin{bmatrix} L & 0 \\ 0 & C \end{bmatrix}}^{\mathbf{a}} \frac{\partial}{\partial t} \overbrace{\begin{bmatrix} I \\ V \end{bmatrix}}^{\mathbf{u}} + \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{\mathbf{b}} \frac{\partial}{\partial x} \begin{bmatrix} I \\ V \end{bmatrix} + \begin{bmatrix} R\,I \\ G\,E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1}$$

Show that the system is hyperbolic, and define its characteristics.

The line properties are henceforth assumed to be uniform in space.

1-b Write the system in normal form.

2-a. To substantiate the nature of the system, obtain the equivalent second order equation that the unknowns I and V satisfy. Observe that this second order equation involves a single variable. Therefore, we have obtained a *decoupled* system, at the price of a higher order operator. This equation is referred to as *telegraph equation.* Find the nature of this system, and comment.

2-b. Consider now a *distortionless* line $RC = LG$. Show that $I(t)\,e^{tR/L}$ and $V(t)\,e^{tR/L}$ satisfy a canonical form of the wave equation, with wave speed $1/\sqrt{LC}$.

When $RC \neq LG$, find a modified function in the same mood as above that satisfies the non homogeneous wave equation.

2-c. So far we have manipulated the equations assuming all line coefficients to be different from zero. Consider now Heaviside's ideal line with $L = G = 0$. What is its nature?

Solution:

1-a. The resulting eigenvalue problem (III.1.12),

$$\boldsymbol{\lambda} \cdot (\mathbf{a} - \mathbf{b}\frac{dt}{dx}) = \mathbf{0}\,, \tag{2}$$

yields two real and distinct eigenvalues dt/dx, and associated independent eigenvectors $\boldsymbol{\lambda}$,

$$\frac{dt_+}{dx} = \sqrt{LC}, \quad \boldsymbol{\lambda}_+ = \begin{bmatrix} \sqrt{C} \\ \sqrt{L} \end{bmatrix}; \quad \frac{dt_-}{dx} = -\sqrt{LC}, \quad \boldsymbol{\lambda}_- = \begin{bmatrix} \sqrt{C} \\ -\sqrt{L} \end{bmatrix}, \tag{3}$$

so that the system is hyperbolic, that is, it is expected to be able to propagate disturbances at finite speed.

1-b Let $\mathbf{d} = \mathrm{diag}[\sqrt{LC}, -\sqrt{LC}]$ be the diagonal matrix of the eigenvalues. We look for a matrix $\mathbf{t}$ such that $\mathbf{t} \cdot \mathbf{a} = \mathbf{d} \cdot \mathbf{t} \cdot \mathbf{b}$, as explained in Exercise III.5. The matrix $\mathbf{t}$ is of course not unique, and in fact there is a double indeterminacy, $t_{12} = t_{11}\sqrt{L/C}$, $t_{21} = -t_{22}\sqrt{C/L}$. We take $t_{11} = 1/(2L\sqrt{C})$, $t_{22} = -1/(2C\sqrt{L})$, and therefore

$$\mathbf{t} = \frac{1}{2\,LC} \begin{bmatrix} \sqrt{C} & \sqrt{L} \\ \sqrt{C} & -\sqrt{L} \end{bmatrix}, \quad \mathbf{t} \cdot \mathbf{a} = \frac{1}{2\sqrt{LC}} \begin{bmatrix} \sqrt{L} & \sqrt{C} \\ \sqrt{L} & -\sqrt{C} \end{bmatrix}. \tag{4}$$

Let us introduce the new unknowns,

$$\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{t} \cdot \mathbf{a} \begin{bmatrix} I \\ V \end{bmatrix} = \frac{1}{2\sqrt{LC}} \begin{bmatrix} \sqrt{L} & \sqrt{C} \\ \sqrt{L} & -\sqrt{C} \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}, \quad \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} \sqrt{C} & \sqrt{C} \\ \sqrt{L} & -\sqrt{L} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}. \tag{5}$$

Upon pre-multiplication by $\mathbf{t}$, the system (1) becomes

$$\frac{\partial}{\partial t} \begin{bmatrix} u \\ v \end{bmatrix} + \frac{1}{\sqrt{LC}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} RC + LG & RC - LG \\ RC - LG & RC + LG \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6}$$

2-a. Applying the operator $-C\partial/\partial t$ to the first line of (1), and to the second line the operator $\partial/\partial x$, adding the results and using again the first line to eliminate the undesirable unknown, we get the telegraph equation,

$$\Big(LC\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + (RC + LG)\,\frac{\partial}{\partial t} + GR\Big)\, X = 0, \quad X = I, V\,. \tag{7}$$

2-b. Let $Y(t) = X(t)\,e^{\alpha\,t}$ with α an unknown exponent. The function $Y(t)$ satisfies the equation,

$$\Big(LC\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + (RC + LG - 2\,\alpha\,LC)\,\frac{\partial}{\partial t} + (\alpha^2\,LC - (RC + LG)\,\alpha + GR)\Big)\, Y = 0\,. \tag{8}$$

The coefficients of the zero and first order terms vanish simultaneously only if $RC = LG$ and then $\alpha = R/L$, and $Y(t)$ satisfies the wave equation,

$$\Big(\frac{\partial^2}{\partial t^2} - \frac{1}{(\sqrt{LC})^2}\,\frac{\partial^2}{\partial x^2}\Big)\, Y = 0\,, \tag{9}$$

where $c = 1/\sqrt{LC}$ appears clearly as the wave speed. A typical value is 3×10^8 m/s. This second order analysis is of course consistent with the hyperbolic nature of the initial first order system.

More generally, if $\alpha = (RC + LG)/(2\,LC)$, the first order term vanishes, and we have an inhomogeneous wave equation,

$$\Big(LC\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{(RC - LG)^2}{4\,LC}\Big)\,Y = 0\,. \tag{10}$$

The solution is then a wave followed by a residual wave due to the source term.

2-c. When $L = G = 0$, the telegraph equation (7) is still valid, but it looses its hyperbolic character and becomes a diffusion equation,

$$\Big(-\frac{\partial^2}{\partial x^2} + RC\,\frac{\partial}{\partial t}\Big)\,X = 0, \quad X = I, V\,, \tag{11}$$

with a diffusion coefficient equal to $1/(RC)$. Therefore in these circumstances, the mode of propagation of the electrical signal is quite different from the general analysis above. For a voltage shock V_0 applied at the end of the line, one might define qualitatively a *beginning of arrival time* at a point x when the voltage is equal to say 10% of V_0, and an *arrival time* when the voltage is say 50% of V_0. As indicated in Chapter I, the solution has the form of the complementary error function, and the characteristic time is in proportion to $RC\,x^2$.